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1988 J. Phys. A: Math. Gen. 21 1103

(http://iopscience.iop.org/0305-4470/21/4/035)

J. Phys. A: Math. Gen. **21** (1988) 1103-1107. Printed in the UK

A mechanism for setting the scale for gauge coupling constants

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Received 29 June 1987

Abstract. We identify the spin connection of four-dimensional spacetime with a subgroup of a simple gauge group G. The remaining subgroups of G are the usual internal symmetry groups. Using the work of Weinberg to relate gauge constants to the circumferences of compact spaces and identifying the known spin of elementary particles with the eigenvalues of the spin connection allows us to set the scale of G. We can then calculate the remaining gauge coupling constants numerically. We show how this mechanism works for the sample calculation where G is SU_4, the spin connection is the usual $SU_2 \times SU_2$ and the internal gauge symmetry is U_1. We find the electromagnetic fine-structure constant to be $\sim \frac{1}{100}$.

1. Introduction

The idea of relating the electromagnetic gauge coupling constant to the circumference of the compact U_1 gauge group goes back to Souriau (1963). He considered the wave equation on the five-dimensional manifold of the Kaluza (1921)-Klein (1926a, b) unification of general relativity and electromagnetism. Chodos and Detweiler (1980) and Gross and Perry (1983) treated the U_1 case similarly. This work had its foundations in work by Bergmann (1942) who showed that the radius of the compact dimension was an invariant of the geometry. Weinberg (1983) extended these ideas to non-Abelian gauge groups and more generally to cases where the compact space is not a group space or a coset space. The gauge group is then the isometry group of the compact space. Weinberg (1983) related the gauge coupling constants to suitably averaged circumferences of the compact space.

It is an exciting thought that one might be able to calculate values for gauge coupling constants using the above ideas. Unfortunately, an arbitrary scale factor for the compact space is always present so that ratios of coupling constants can be calculated but not their numerical values. This dilatation invariance was pointed out by Gross and Perry (1983) in the electromagnetic case. Appelquist and Chodos (1983) attempted to break the dilatation invariance using quantum corrections but were unable to get a finite result. The fifth dimension underwent collapse over the range of validity of their calculation.

In the present paper we try a new approach to setting the scale of gauge groups. We use the idea of setting the spin connection in a manifold equal to a subgroup of the gauge group. Candelas *et al* (1985) did this in the context of finding a solution for the vacuum configurations of superstrings. Wilczek (1977) and Charap and Duff (1977) also set the vector potential equal to the spin connection as a way of finding solutions of the O(4) Yang-Mills gauge field equations by using known solutions in general relativity.

For the present work, we start with a simple gauge group large enough to contain the usual gauge groups of interest as subgroups plus a subgroup which can be set equal to the spin connection of the base manifold, here taken to be a general four-dimensional Riemannian space of general relativity. Since one subgroup is set equal to the spin connection, we use the known minimum non-zero value for the quantised spin of a particle, $\frac{1}{2}\hbar$, to set the scale for this subgroup. This also sets the scale for all the other subgroups, since Weinberg's approach, or simple group theory, fixes the ratios of the usual gauge coupling constants associated with the other subgroups to the 'gauge coupling constant' associated with the subgroup which was identified with the spin connection. We demonstrate this mechanism for the simple case where the spin connection is the usual $SU_2 \times SU_2$ and the gauge group of interest is the U_1 of electromagnetism. The simple group is SU_4, and we find $\alpha_{em} \sim \frac{1}{100}$.

2. Spin connection

Let us now turn to the details of the calculation. To carry out our program, we need the spin connection of our base space. The base space will be taken to be the four-dimensional Riemannian geometry of general relativity so that the spin connection refers to the spin of ordinary particles. Also we are primarily interested in local physics below. Thus we deal only with the infinitesimal holonomy group. The infinitesimal holonomy group (IHG) is the group generated by the linear connection in the tangent space (Loos 1967). It has been used to classify four-dimensional Riemannian spaces by Schell (1961) and Goldberg and Kerr (1961). This linear connection is the spin connection so that, on a manifold with a given IHG, the spin connection can be considered to be a gauge field of this same group. In particular, in four dimensions, the IHG is the holonomy group associated with rotations in the four-dimensional tangent space. If the signature is $(++++)$, the IHG is a subgroup of O(4) (Candelas *et al* 1985, Lichnerowicz 1976). If the signature is $(+++-)$, the IHG is a subgroup of O(3, 1). Using the usual unitary trick of $t \to it$, the Lorentz group SO(3, 1) becomes SO(4, R). Now spinors involve a two-to-one covering of the tangent space. The IHG is unaffected by this covering since a two-to-one pullback is involved. Then $\text{Spin}(4) \equiv SU_2 \times SU_2$ is the simply connected covering group of SO(4, R). (The representations of SO(4, R) correspond to non-unitary representations of the non-compact SO(3, 1), but unitarity is regained in the physical solutions of the Dirac equation.) We can also consider SO(4, c) which has both SO(4, R) and SO(3, 1) as subgroups. The simply connected covering group of SO(4, c) is $\text{Sl}(2, c) \times \text{Sl}(2, c)$ which has $SU_2 \times SU_2$ as its maximal compact subgroup (Hermann 1975). Now, fibre bundles are the natural language to use in describing gauge theories. We note that a principal bundle with structure group $\text{Sl}(2, c) \times \text{Sl}(2, c)$ is equivalent to a principal bundle with structure group $SU_2 \times SU_2$ (Nash and Sen 1983). We will take the spin connection appropriate for the description of spinors in physics to be $SU_2 \times SU_2$ from the above considerations. This is, of course, a well known result. It agrees with the results of Schell (1961) who found that for four-dimensional spacetime a six-parameter IHG is needed in general if all Petrov types are to be allowed. We note that having a spinor structure exist globally involves the Stiefel-Whitney classes (Milner 1963, Bichteler 1968, Geroch 1968, 1970).

3. Sample calculation using the scale setting mechanism

We now want to do a sample calculation to show how our mechanism for setting the scale of gauge coupling constants works. We consider a world in which the only usual gauge interaction present is electromagnetism. We take the gauge group to be SU_4 and identify the $SU_2 \times SU_2$ subgroup with the $SU_2 \times SU_2$ spin connection discussed above. The U_1 subgroup refers to electromagnetism. This identification of the $SU_2 \times SU_2$ spin connection of the base space with an $SU_2 \times SU_2$ subgroup of the gauge group is crucial and profound. It relates a spacetime symmetry to a gauge symmetry by treating spin in some sense as a gauge field like electromagnetism. This is reasonable since Weinberg (1983) has shown that gauge coupling constants are associated with compact gauge groups. In some sense spin, which comes in units of $\hbar/2$, is a 'charge' associated with $SU_2 \times SU_2$ just like electric charge is associated with U_1.

Before we find the relationship between the $SU_2 \times SU_2$ and the U_1 coupling constants, we need the diagonal generators of SU_4 in a representation which displays the $SU_2 \times SU_2$ subgroup structure. The 15 traceless Hermitian generators can be taken to be

$$L^{1,2,3} = \begin{bmatrix} \sigma_{1,2,3} & & 0 & 0 \\ & & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad L^{4,5,6} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & \sigma_{1,2,3} & \end{bmatrix} \tag{1}$$

where σ_i are the 2×2 Pauli spin matrices.

If we define a matrix

$$\begin{bmatrix} 0 & 0 & A & B \\ 0 & 0 & C & D \\ A^* & C^* & 0 & 0 \\ B^* & D^* & 0 & 0 \end{bmatrix}$$

then L^7, L^{11} correspond to $A = -\mathrm{i}, 1$ with $B = C = D = 0$; L^8, L^{12} correspond to $B = -\mathrm{i}, 1$ with $A = C = D = 0$; L^9, L^{13} corresponds to $C = -\mathrm{i}, 1$ with $A = B = D = 0$ and L^{10}, L^{14} correspond to $D = -\mathrm{i}, 1$ with $A = B = C = 0$. These are analogous to the SU_3 matrices of Gell-Mann (1962). To find the diagonal generator L^{15} we use

$$\mathrm{Tr}(L^i L^j) = 2\delta_{ij} \tag{2}$$

for normalisation and also the expression

$$\mathrm{Tr}\, L^k [L^i, L^j] = 4\mathrm{i} f^{ijk} \tag{3}$$

where the f^{ijk} are the structure constants and must be totally antisymmetric (Gell-Mann 1962). We find

$$L^{15} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}. \tag{4}$$

The other diagonal generators are

$$L^3 = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \qquad L^6 = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \tag{5}$$

from above.

Weinberg (1983) considers a compact space and an isometry group $\mathscr{D}$ of that space. t_α are Hermitian generators of $\mathscr{D}$ and e^α a vector satisfying

$$\delta_{\alpha\beta}e^\alpha e^\beta = 1. \qquad (6)$$

He shows that, if for some e^α all eigenvalues of $e^\alpha t_\alpha$ are integer multiples of the eigenvalue g_e of lowest non-zero absolute value, then g_e plays the role of the gauge coupling constant and is given by

$$g_e = 2\pi\kappa/\mathscr{C}N_e \qquad (7)$$

where the representation used is N_e valued for the subgroup generated by $e^\alpha t_\alpha$, $\kappa = (16\pi G)^{1/2}$ and $\mathscr{C}$ is an appropriate root mean square circumference of the compact manifold. We will take the gauge space of SU_4 itself for the compact manifold. If we look at the subgroup generated by $e_\alpha L^\alpha$ where L^α are the Hermitian generators of SU_4 and let $e_3 = 1$ and all other $e_\alpha = 0$, we find

$$g_{SU_2} = (1)\times\kappa/R \qquad (8)$$

where R is an arbitrary scale factor for the SU_4 group space and we are using units with $\hbar = c = 1$. Letting $e_6 = 1$ and all other $e_\alpha = 0$ gives the same result for the other SU_2 subgroup. The 1 in (8) arises from the form of the generators in (5). If we let $e_{15} = 1$ and all other $e_\alpha = 0$, we get, using (4), that

$$g_{U_1} = \frac{1}{\sqrt{2}}\frac{\kappa}{R} \qquad (9)$$

where the same scale factor appears in both (8) and (9), since we have a simple group.

We now use the fact that we identify the $SU_2\times SU_2$ spin connection with the $SU_2\times SU_2$ subgroup of SU_4. This means that g_{SU_2} in (8) refers to the smallest unit of spin, namely $\hbar/2$. In our above units with $\hbar = 1$ we then have

$$g_{SU_2} = \tfrac{1}{2} = \kappa/R. \qquad (10)$$

Equation (10) now sets the scale for R. We are using $\hbar$ and its relation to spin to get the gauge group scale. Once the scale of R is set, we can use it to calculate the remaining gauge coupling constants. In the present simple example, putting (10) into (9) gives

$$g_{U_1} \equiv (4\pi\alpha_{em})^{1/2} = 1/2\sqrt{2} \qquad (11)$$

and the electromagnetic fine-structure constant becomes

$$\alpha_{em} = 1/32\pi \simeq \tfrac{1}{100}. \qquad (12)$$

We note that the numerical constants in (8) and (9) arise from the structure of the simple gauge group, SU_4 in the present case. Equation (10) which sets the scale is a direct consequence of equating the $SU_2\times SU_2$ spin connection with the $SU_2\times SU_2$ subgroup of the gauge group.

To calculate α_{em} in the real world, we need a much larger gauge group which includes the other interactions of nature. String theory (Schwarz 1982, Green 1983, Gross *et al* 1985a, b) may provide such a group. A group like SO_{14} is also a possibility. This group must have an $SU_2\times SU_2$ subgroup which we can identify with the spin connection as above plus a subgroup large enough to contain all the 'internal' gauge symmetries. We also need to know the symmetry breaking pattern and need to include the effects of running coupling constants. The coupling constants we calculate will apply at some high energy scale. In spite of these difficult problems, nonetheless, the above mechanism can be used to set the scale of the gauge group and should ultimately allow a realistic calculation of all the gauge coupling constants in the problem.

Acknowledgment

I am indebted to Derek Pursey for many informative discussions concerning this paper.

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